

Nonlinear differential equations with exact solutions

N.A. Kudryashov

Department of Applied Mathematics
Moscow Engineering and Physics Institute
(State university)
31 Kashirskoe Shosse, 115409
Moscow, Russian Federation

Abstract

New problem is considered that is to find nonlinear differential equations with special solutions. Method is presented to construct nonlinear ordinary differential equations with exact solution. Crucial step to the method is the assumption that nonlinear differential equations have exact solution which is general solution of the simplest integrable equation. The Riccati equation is shown to be a building block to find a lot of nonlinear differential equations with exact solutions. Nonlinear differential equations of the second, third and fourth order with special solutions are given. Most of these equations are used at the description of processes in physics and in theory of nonlinear waves.

Keywords: Riccati equation, nonlinear differential equation, exact solution

PACS: 02.30.Hq - Ordinary differential equations

1 Introduction

In recent years one can observe a splash of papers where authors presented a lot of different approaches to look for exact solutions of nonlinear differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9]. There are two reasons to make the study in this direction. First there is a great interest to the investigation of nonlinear phenomenon. Secondly we have codes Maple, Mathematica and other ones to conduct a lot of symbolical calculations.

It is well known that all nonlinear differential equations can be connectionally divided into three types: exactly solvable, partially solvable and those that have no exact solution.

Consider nonlinear evolution equation

$$E_1[u] \equiv E_1(u, u_t, u_x, \dots, x, t) = 0 \quad (1.1)$$

Assume we need to have exact solutions of this equation. First of all one can try to solve a question about the integrability of this equation. For this aim one

can apply the Painlevé test to check the integrability of this equation [10]. As a result we can have two variants. First variant is the equation (1.1) pass the Painlevé test and we have the necessary condition for integrability of equation in this case. Second variant is the equation (1.1) does not pass the Painlevé test. In this case we can look for some transformation to obtain equation that can be passed the Painlevé test. However we often obtain that the equation (1.1) does not pass the Painlevé test and we have not got any transformation to obtain good form for the origin equation. Unfortunately a lot of nonlinear evolution equation does not pass the Painlevé test. In any case one can search exact solutions of nonlinear differential equation. In fact one can look for exact solutions of nonlinear differential equation of all types without solution of the problem about the integrability.

Usually we look for exact solution of nonlinear evolution equation taking into account the travelling wave and search exact solution of equation (1.1) in the form

$$u(x, t) = y(z), \quad z = x - C_0 t \quad (1.2)$$

As a result we have that the equation (1.1) reduces to the nonlinear ordinary differential equation (ODE)

$$E_2[y] \equiv E_2(y, y_z, \dots, z) = 0 \quad (1.3)$$

To obtain exact solutions of equation (1.3) one can apply different approaches [2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. However one can note that the most methods that are used to search exact solutions take into account the singular analysis for solutions of the nonlinear differential equations [10, 11, 12, 21, 22, 23]. Even though the investigators are not aware of the analytical theory of nonlinear differential equations they apply as a rule its ideas and approaches.

Using the singular analysis first of all we have to consider the leading members of equation (1.3). After that we find the singularity for solution of equation (1.3). Further the truncated expansion is used to have the transformation to search exact solutions of nonlinear ODEs. At this point one can use some trial functions (hyperbolic and elliptic and so on) to look for exact solutions of nonlinear ODEs [2, 3, 24, 25, 26, 27, 28, 29, 30].

However one can note that hyperbolic and elliptic functions are general solutions of nonlinear exactly solvable equations. We have as a rule that partially solvable nonlinear differential equations have exact solutions that are general solutions of solvable equations of lesser order.

In this paper we are going to start the solution of the new problem that is to find nonlinear ordinary differential equations of polynomial form which have special solutions. Let us explain the idea of our work. It is well known there is the great problem that is to find nonlinear differential equations that are integrable. It is very important because we often want to have general solution of nonlinear ordinary equations. For a example Painlevé and his school found 50 canonical forms of the second order class for the ordinary differential equations that are integrable equations. However sometimes we can content ourselves with some special solutions because a lot of differential equations are nonintegrable ones although they are intensively used in physics and look as simple equations.

It is important to find some special solutions of these equations that are called exact solutions. In this connection it is convenient to have the list of nonlinear differential equations that are not integrable but have special solutions.

The aim of this work is to present new method to find nonlinear differential equations with exact solutions. Using our approach we give a class of nonlinear ordinary differential equations that have exact solutions. Special solutions of our nonlinear ordinary differential equations are found via general solution of the Riccati equation.

As this takes place we do not solve the problem of finding all possible exact solutions for our class of nonlinear ordinary differential equations. Our class contains a lot of exactly solvable equations. In this paper we give only the polynomial class of nonlinear ordinary differential equations with special solutions.

The outline of this paper is as follows. In section 2 we present the method to find nonlinear ordinary differential equations (ODEs) with special solutions. Nonlinear ODEs with exact solutions of the first, second, third and fourth degree singularities are given in section 3, 4, 5 and 6. Example of nonlinear ODE with exact solution of the fifth degree singularity is considered in section 7.

2 Method applied

Let us discuss the method that can be applied to find nonlinear differential equations with exact solutions. One can note that most nonlinear ordinary differential equations has exact solutions that are general solutions of differential equations of lesser order. Much more than that exact equations for the most nonlinear differential equations are determined via general solution of the Riccati equation. This is so indeed because one can note that the most approaches to search exact solutions of nonlinear ordinary differential equations are based on general solution of the Riccati equation. The application of the tanh method confirms this idea [24, 28, 29, 31, 32]. This is due to the fact that the Riccati equation is simplest equation of first order of polynomial form that have the Painleve property [10, 17, 21].

The Riccati equation can be presented in the form

$$R[Y] = Y_z + Y^2 + q(z) = 0 \quad (2.1)$$

We have the following simple theorem.

Theorem 2.1. *Let $Y(z)$ be solution of equation (2.1) than equations*

$$Y_{zz} = 2Y^3 + 2qY - q_z \quad (2.2)$$

$$Y_{zzz} = -6Y^4 - 8qY^2 + 2q_zY - 2q^2 - q_{zz} \quad (2.3)$$

$$Y_{zzzz} = 24Y^5 + 40qY^3 + 16q^2Y - 10q_zY^2 + 2q_{zz}Y - 6qq_z - q_{zzz} \quad (2.4)$$

have special solutions that are expressed via general solution of equation (2.1)

Proof. Theorem 2.1 is proved by differentiation of (2.1) with respect to z and substitution Y_z from equation (2.1) and so on into expressions obtained. \square

Corollary 2.1. Equations

$$Y_{zz} = 2Y^3 - 2\alpha Y \quad (2.5)$$

$$Y_{zzz} = -6Y^4 + 8\alpha Y^2 - 2\alpha^2 \quad (2.6)$$

$$Y_{zzzz} = 24Y^5 - 40\alpha Y^3 + 16\alpha^2 Y \quad (2.7)$$

have special solutions $Y(z)$ where $Y(z)$ is the general solutions of the Riccati equation in the form

$$Y_z + Y^2 - \alpha = 0 \quad (2.8)$$

It is well known that general solutions of equation (2.8) is

$$Y(z) = \sqrt{\alpha} \tanh(\sqrt{\alpha}z + \varphi_0) \quad (2.9)$$

where φ_0 is arbitrary constant.

We want to find nonlinear ordinary differential equation which have special solutions that are determined via general solution of the Riccati equation.

Algorithm of our method can be presented by *four steps*. *At the first step* we choose the singularity of special solution and give the form of this solution. *At the second step* we take the order of nonlinear ordinary differential equation what we want to search. *The third step* lies in the fact that we write the general form of nonlinear differential equation taking into account the singularity of the solution and the given order for nonlinear differential equation. *The fourth step* contains calculations. As a result we find limitations for the parameters in order for nonlinear differential equation has exact solutions. At this step we have nonlinear ODE with exact solutions.

Let us demonstrate our approach. With this aim let us find a nonlinear ordinary differential equation of the second order with solution of the second degree singularity. This solution takes the form

$$y(z) = A_0 + A_1 Y + A_2 Y^2 \quad (2.10)$$

Here $Y(z)$ is a solution of equation (2.8). Without loss of generality we can take $A_2 = -12$.

First of all let us write the general form of the nonlinear second order ordinary differential equation with solution (2.10). It takes the form

$$E_1 \equiv y_{zz} + a_0 y^2 - b_0 y_z - C_0 y + C_1 = 0 \quad (2.11)$$

Here a_0, b_0, C_0 and C_1 are unknown parameters of equation

This equation was written using singularity of solution (2.10). Actually the singularity of the first term in (2.11) is equal to 4 and for the polynomial form of the equation we have to take into account the same singularity of nonlinear expression for the second term (2.11). Other terms in (2.11) have lesser singularity.

We want to find equation (2.11) that has exact solution (2.9). We can do this if we define values of parameters a_0, b_0, C_0 and C_1 . Substituting (2.10) at $A_2 = -12$ into equation (2.11) and taking into account equations (2.5) and (2.8) we have equation in the form

$$\begin{aligned}
& (-72 + 144 a_0) Y^4 + (-24 a_0 A_1 + 2 A_1 - 24 b_0) Y^3 + \\
& + (-24 a_0 A_0 + a_0 A_1^2 + 12 C_0 + b_0 A_1 + 96 \alpha) Y^2 + \\
& + (-C_0 A_1 - 2 A_1 \alpha + 2 a_0 A_0 A_1 + 24 b_0 \alpha) Y - \\
& - 24 \alpha^2 + a_0 A_0^2 - C_0 A_0 - b_0 A_1 \alpha + C_1 = 0
\end{aligned} \tag{2.12}$$

Equating expressions in equation (2.12) at different degrees of Y to zero we have algebraic equations for the parameters a_0, a_1, b_0 and C_1 in the form

$$a_0 = \frac{1}{2} \tag{2.13}$$

$$A_1 = -\frac{12}{5} b_0 \tag{2.14}$$

$$A_0 = \frac{1}{25} b_0^2 + C_0 + 8 \alpha \tag{2.15}$$

$$\alpha = \frac{1}{100} b_0^2 \tag{2.16}$$

$$C_1 = -\frac{18}{625} b_0^4 + \frac{1}{2} C_0^2 \tag{2.17}$$

As a result we obtain the nonlinear second order differential equation in the form

$$y_{zz} + \frac{1}{2} y^2 - b_0 y_z - C_0 y - \frac{18}{625} b_0^4 + \frac{1}{2} C_0^2 = 0 \tag{2.18}$$

Exact solution of equation (2.18) is determined by the formula

$$y(z) = \frac{3}{25} b_0^2 + C_0 - \frac{12}{5} b_0 Y - 12 Y^2 \tag{2.19}$$

Where $Y(z)$ is a solution of the Riccati equation

$$Y_z = -Y^2 + \frac{b_0^2}{100} \quad (2.20)$$

which is

$$Y(z) = \pm \frac{b_0}{10} \tanh \left(\pm \frac{b_0 z}{10} + \varphi_0 \right) \quad (2.21)$$

Here φ_0 is arbitrary constant. Actually we find equation (2.18) with fascinating history. The matter is this equation can be found from the Korteveg – de Vries – Burgers equation which takes the form

$$u_t + uu_x + u_{xxx} = b_0 u_{xx} \quad (2.22)$$

Equation (2.22) is a generalization of the Korteveg – de Vries equation in the case of dissipative processes [33]. Equation (2.22) in contradistinction to the Korteveg – de Vries equation is not integrable equation.

Using the travelling wave (1.2) we have from (2.22) after integration over z the following equation

$$C_1 - C_0 y + \frac{1}{2} y^2 + y_{zz} - b_0 y_z = 0 \quad (2.23)$$

We can see that equation (2.23) is equation (2.18) as well.

However exact solutions of equation (2.23) were unknown over many years. We hope that exact solutions (2.19) of equation (2.17) were found first in the work [11]. However this exact solution have rediscovered in a number of papers [34, 35, 36].

Assuming $b_0 = 0$ in equation (2.18) we have exactly solvable equation solution. This one can be found via the elliptic function. Nonlinear evolution equation (2.22) in this case takes the form of the Korteveg – de Vries equation and solution (2.19) is the soliton.

One can see we can obtain nonlinear integrable ODEs too using our approach.

We have to note that equation (2.18) can be transformed to nonlinear ODE that is found from the nonlinear evolution equation

$$u_t = u_{xx} - C_0 u - d_0 u^2 \quad (2.24)$$

This is the Fisher equation [37] or the Kolmogorov – Petrovskii – Piskunov equation [38]. Special solutions of this equation were found first in the work [39] and repeated a lot of times later.

3 Nonlinear ODEs with exact solutions of the first degree singularity

Taking into account nonlinear ODEs (2.5) – (2.8) one can find nonlinear differential equations with exact solutions. These exact solutions are expressed via the general solution of the Riccati equation

$$Y_z = -Y^2 + \alpha \quad (3.1)$$

First of all let us assume that nonlinear ODEs have special solutions of the first degree singularity

$$y(z) = A_0 + A_1 Y(z) \quad (3.2)$$

where $Y(z)$ is a solution of the equation (3.1).

Without loss of generality we can assume $A_0 = 0$ and $A_1 = 1$.

3.1. Second order ODEs. Let us find the nonlinear second order ODEs with exact solutions (3.2) where $Y(z)$ is a solution of equation (3.1).

Second order ODEs with solutions of the first degree singularity can take the form

$$y_{zz} + a_0 y y_z + a_1 y^3 - b_0 y_z + b_1 y^2 - \alpha C_0 y + \alpha C_1 = 0 \quad (3.3)$$

Here a_0 , a_1 , b_0 , b_1 , C_0 and C_1 are unknown coefficients of equation (3.3). Our problem is to find some relations between coefficients of equation (3.3) so that equation (3.3) has exact solution (3.2).

Substitution of (3.2) and Eqs. (3.1) and (2.5) into equation (3.3) lead to relation

$$(a_1 - a_0 + 2)Y^3 + (b_1 + b_0)Y^2 + \alpha(a_0 - C_0 - 2)Y + \alpha C_1 - \alpha b_0 = 0 \quad (3.4)$$

From equation (3.4) we have

$$\begin{aligned} a_1 &= a_0 - 2, \\ b_1 &= -b_0, \\ a_0 &= C_0 + 2, \\ b_0 &= C_1 \end{aligned} \quad (3.5)$$

As a result we obtain the second order ODEs with exact solutions (3.2) at $A_0 = 0$ and $A_1 = 1$ in the form

$$y_{zz} + ((2 + C_0)y - C_1) y_z + C_0 y^3 - C_1 y^2 - \alpha C_0 y + \alpha C_1 = 0 \quad (3.6)$$

Here α is arbitrary constant.

At $C_0 = -2$ we get ODE

$$y_{zz} - C_1 y_z - 2y^3 - C_1 y^2 + 2\alpha y + \alpha C_1 = 0 \quad (3.7)$$

This equation can be found from the modified Korteweg – de Vries – Burgers equation

$$u_t - (2C_1 u + 6u^2) u_x + u_{xxx} = C_1 u_{xx} \quad (3.8)$$

using the travelling wave (1.2). We have found that equation (3.8) has solution (3.2). Exact solutions of equation was found in work [40].

3.2. Third order ODEs. Now let us find the nonlinear third order ordinary differential equations with exact solutions (3.2).

Nonlinear third order ordinary differential equations of the polynomial type can have exact solution of the first degree singularity if this one takes the form

$$\begin{aligned} a_0 y_{zzz} + a_1 y y_{zz} + a_2 y^2 y_z + a_3 y z^2 + a_4 y^4 + b_0 y_{zz} + b_1 y y_z + b_2 y^3 + \\ + d_0 y_z + d_1 y^2 - \alpha C_0 y + \alpha C_1 = 0 \end{aligned} \quad (3.9)$$

Substitution (3.2), (3.1), (2.5) and (2.6) into equation (3.9) leads to the equation

$$\begin{aligned} (a_4 - a_2 + a_3 - 6a_0 + 2a_1) Y^4 + (2b_0 + b_2 - b_1) Y^3 + \\ + (-a_2\alpha + d_1 - d_0 + 2a_3\alpha + 2a_1\alpha - 8a_0\alpha) Y^2 + \\ + (2b_0\alpha - b_1\alpha - \alpha C_0) Y + \alpha C_1 - 2a_0\alpha^2 - d_0\alpha + a_3\alpha^2 = 0 \end{aligned} \quad (3.10)$$

We have from equation (3.10)

$$a_4 = a_2 + 6a_0 - a_3 - 2a_1 \quad (3.11)$$

$$b_2 = -C_0 \quad (3.12)$$

$$d_1 = d_0 + \alpha(a_2 - 2a_3 + 8a_0 - 2a_1) = 0 \quad (3.13)$$

$$b_1 = 2b_0 - C_0 = 0 \quad (3.14)$$

$$\alpha = \frac{d_0 - C_1}{a_1 - 2a_0} \quad (3.15)$$

As a result we obtain the nonlinear third order differential equation in the form

$$\begin{aligned} & a_0 y_{zzz} + (a_1 y + b_0) y_{zz} + a_3 y_z^2 + (a_2 y^2 + 2 y b_0 - y C_0 + d_0) y_z + \\ & + (a_2 - a_3 + 6 a_0 - 2 a_1) y^4 - C_0 y^3 + \\ & + (a_2 \alpha - 2 a_3 \alpha + d_0 + 8 a_0 \alpha - 2 a_1 \alpha) y^2 - \alpha C_0 y + \alpha C_1 = 0 \end{aligned} \quad (3.16)$$

Solution of equation (3.16) is found by formula (3.1) at $A_0 = 0$ and $A_1 = 1$. Assuming $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $b_0 = C_0/2$ in equation (3.16) we get

$$a_0 y_{zzz} + \frac{1}{2} C_0 y_{zz} + d_0 y_z + 6 a_0 y^4 - C_0 y^3 + (8 \alpha a_0 + d_0) y^2 - \alpha C_0 y + \alpha C_1 = 0 \quad (3.17)$$

This equation can be found from the generalized Kuramoto – Sivashinsky equation [11, 17]

$$u_t + \varepsilon u^3 u_x + \beta u_{xx} + \gamma u_{xxx} + \delta u_{xxxx} = 0 \quad (3.18)$$

if we look for solution of equation (3.18) using the travelling wave (1.2). In this case we have from equation (3.18)

$$C_1 - C_0 y + \frac{\varepsilon}{4} y^4 + \beta y_z + \gamma y_{zz} + \delta y_{zzz} = 0 \quad (3.19)$$

One can see that equation (3.17) and (3.18) have exact solutions that are expressed via general solution of the Riccati equation

3.3. Fourth order ODEs. General form of nonlinear ordinary differential equation of the fourth order can be written as the following

$$\begin{aligned} & a_0 y_{zzzz} + a_1 y y_{zzz} + a_2 y^2 y_{zz} + a_3 y_z y_{zz} + a_4 y y_z^2 + a_5 y^3 y_z + a_6 y^5 + \\ & + b_0 y_{zzz} + b_1 y y_{zz} + b_2 y^2 y_z + b_3 y_z^2 + b_4 y^4 + h_0 y_{zz} + h_1 y y_z + \\ & + h_2 y^3 + d_0 y_z + d_1 y^2 - C_0 \alpha y + C_1 \alpha = 0 \end{aligned} \quad (3.20)$$

Substituting equation (2.5) – (2.8) into equation (3.20) as a result we have nonlinear fourth order ordinary differential equation with exact solutions that is

determined via general solution of the Riccati equation (2.8). It takes the form

$$\begin{aligned}
& a_0 y_{zzzz} + (a_1 y + b_0) y_{zzz} + (b_1 y + a_3 y_z + a_2 y^2 + h_0) y_{zz} + \\
& + (a_4 y + b_3) y_z^2 + \\
& + (d_0 + a_5 y^3 + y C_0 + b_2 y^2 - y a_4 \alpha + 2 y a_1 \alpha - 16 y a_0 \alpha) y_z + \\
& + (2 y a_3 \alpha + 2 y h_0) y_z + (-a_4 + 2 a_3 + 6 a_1 - 24 a_0 - 2 a_2 + a_5) y^5 + \\
& + (-b_3 - 2 b_1 + b_2 + 6 b_0) y^4 + \\
& + (-a_5 \alpha + a_4 \alpha + 2 a_2 \alpha - 6 a_1 \alpha - 2 a_3 \alpha + 24 a_0 \alpha + C_0) y^3 + \\
& + (2 b_3 \alpha - b_2 \alpha - 8 b_0 \alpha + d_0 + 2 b_1 \alpha) y^2 - C_0 \alpha y + C_1 \alpha = 0
\end{aligned} \tag{3.21}$$

Here α is determined by the formula

$$\alpha_1 = 0, \quad \alpha_2 = \frac{d_0 + C_1}{2b_0 - b_3} \tag{3.22}$$

From equation (3.22) one can find some nonlinear ODE which are useful in physics.

Assuming $a_0 = 1$, $a_1 = -10$, $a_2 = 0$, $a_3 = -10$, $a_4 = 0$, $a_5 = 6$, $b_0 = b_1 = b_2 = b_3 = b_4 = d_0 = d_1 = g_0 = g_1 = d_2 = 0$ in equation (3.21) we have nonlinear ODE in the form

$$y_{zzzz} - 10y^2 y_{zz} - 10y y_z^2 + 6y^5 - \alpha C_0 y + \alpha C_1 = 0 \tag{3.23}$$

This is integrable equation and is found from the modified Korteweg – de Vries equation of the fifth order [41]

$$u_t + \frac{\partial}{\partial x} (u_{xxxx} - 10u^2 u_{xx} - 10u u^2 + 6u^5) = 0 \tag{3.24}$$

Using the travelling wave (1.2) from (3.24) we have equation (3.23).

We can see that equation (3.23) is relative to the class of nonlinear ODEs (3.21). Special solution of equation is determined by the formula (3.2) at $A_0 = 0$, $A_1 = 1$, $C_0 = 6\alpha$ and $C_1 = 0$.

Assuming $a_0 = 1$, $a_1 = -5$, $a_2 = 5$, $a_3 = -5$, $a_4 = 0$, $a_5 = 1$, $b_0 = b_1 = b_2 = b_3 = b_4 = d_0 = d_1 = d_2 = g_0 = g_1 = 0$ in (3.21) we have equation in the form [42, 43]

$$y_{zzzz} - 5y^2 y_z + 5y_z y_{zz} - 5y y_z^2 + y^5 - \alpha C_0 y + \alpha C_1 = 0 \tag{3.25}$$

This is integrable equation too. This one can be found from the Fordy – Gibbons equation

$$u_t + \frac{\partial}{\partial x} (u_{xxxx} - 5u^2u_x + 5u_xu_{xx} - 5uu_x^2 + u^5) = 0 \quad (3.26)$$

if we look for solution in the travelling wave (1.2).

We get that equation (3.25) has special solution (3.2) at $A_0 = 0$, $A_1 = 1$, $C_0 = \alpha$ and $C_1 = 0$.

We can see that a number of integrable equations are abundant in the class of nonlinear ordinary differential equations (3.21).

Assuming $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $b_3 = 0$, $b_2 = 0$, $b_1 = 0$, $h_0 = -\frac{C_0}{2} + 8\alpha a_0$ we have

$$\begin{aligned} & a_0 y_{zzzz} + b_0 y_{zzz} + \left(-\frac{1}{2}C_0 + 8a_0\alpha\right) y_{zz} + d_0 y_z + \\ & + (-a_4 + 2a_3 + 6a_1 - 24a_0 - 2a_2 + a_5) y^5 + \\ & + (-b_3 - 2b_1 + b_2 + 6b_0) y^4 + \end{aligned} \quad (3.27)$$

$$+ (-a_5\alpha + a_4\alpha + 2a_2\alpha - 6a_1\alpha - 2a_3\alpha + 24a_0\alpha + C_0) y^3 +$$

$$+ (2b_3\alpha - b_2\alpha - 8b_0\alpha + d_0 + 2b_1\alpha) y^2 - C_0\alpha y + C_1\alpha = 0$$

This equation can be found from the nonlinear evolution equation of fifth order that takes the form

$$\begin{aligned} & u_t + (\alpha u + \beta u^2 + \gamma u^3 + \delta u^4) u_x + d_0 u_{xx} + \left(8\alpha a_0 - \frac{1}{2}C_0\right) u_{xxx} + \\ & + b_0 u_{xxxx} + a_0 u_{xxxxx} = 0 \end{aligned} \quad (3.28)$$

Some variants of this equation are used at description of nonlinear waves [44, 45, 46, 47].

4 Nonlinear ODEs with exact solutions of the second degree singularity

Let us find nonlinear ordinary differential equations with exact solutions

$$y(z) = A_0 + A_1 Y + A_2 Y^2 \quad (4.1)$$

where $Y(z)$ is a solution of the Riccati equation as well.

Second order class with solutions (4.1) we found before in Introduction. Now let us find the third order class of nonlinear ODEs with solutions (4.1).

4.1. Third order ODEs. Nonlinear third order ODEs with solution (4.1) can have the form

$$a_0 y_{zzz} + a_1 y y_z + b_0 y_{zz} + b_1 y^2 + d_1 y - C_0 y + C_1 = 0 \quad (4.2)$$

Substitution (4.1) into equation (4.2) lead to the algebraic equations for the coefficients of equation (4.2). As a result of calculations at $A_2 = -12$, $A_1 = 12\beta$, $A_0 = 0$ we have

$$a_1 = a_0 \quad (4.3)$$

$$b_0 = 2b_1 + 5\beta a_0 \quad (4.4)$$

$$d_0 = (\beta^2 + 8\alpha)a_0 + 10\beta b_1 \quad (4.5)$$

$$b_1 = -\frac{C_0 - \beta^3 a_0 + 4\alpha\beta a_0}{2(\beta^2 + 8\alpha)} \quad (4.6)$$

$$\alpha_1 = \frac{1}{4}\beta^2, \quad \alpha_2 = -\frac{C_0}{12\beta a_0} \quad (4.7)$$

Using values for α from (4.7) we get

$$b_1^{(1)} = -\frac{C_0}{6\beta^2}, \quad b_2^{(2)} = \frac{1}{2}\beta a_0 \quad (4.8)$$

$$d_0^{(1)} = 3\beta^2 a_0 - \frac{5C_0}{3\beta}, \quad d_0^{(2)} = 6\beta^2 a_0 - \frac{2C_0}{3\beta} \quad (4.9)$$

$$b_0^{(1)} = 5\beta a_0 - \frac{C_0}{3\beta^2}, \quad b_0^{(2)} = 6\beta a_0 \quad (4.10)$$

$$C_1^{(1)} = \frac{9}{2}\beta^2 C_0, \quad C_1^{(2)} = 6\beta^2 C_0 + \frac{C_0^2}{2\beta a_0} \quad (4.11)$$

We have two nonlinear ODEs with special solutions

$$\begin{aligned} & a_0 y_{zzz} + a_0 y y_z + \left(5\beta a_0 - \frac{C_0}{3\beta^2}\right) y_{zz} - \\ & - \frac{C_0 y^2}{6\beta^2} + \left(3\beta^2 a_0 - \frac{5C_0}{3\beta}\right) y_z - C_0 y + \frac{9\beta^2}{2} C_0 = 0 \end{aligned} \quad (4.12)$$

and

$$a_0 y_{zzzz} + a_0 y y_z + 6\beta a_0 y_{zz} + \frac{1}{2}\beta a_0 y^2 +$$

$$+ \left(6\beta^2 a_0 - \frac{2C_0}{3\beta}\right) y_z - C_0 y + \frac{C_0^2}{2\beta a_0} + 6\beta^2 C_0 = 0 \quad (4.13)$$

Solutions of equations (4.12) and (4.13) are found by the formula

$$y(z) = 12\beta Y - 12Y^2 \quad (4.14)$$

where $Y(z)$ is a solution of the Riccati equations

$$Y_z = -Y^2 + \frac{1}{4}\beta^2 \quad (4.15)$$

and

$$Y_z = -Y^2 - \frac{C_0}{12\beta a_0} \quad (4.16)$$

Equations (4.12) and (4.13) are obtained from nonlinear evolution equation

$$u_t + \lambda_1 u u_x + \lambda_2 u_{xxx} + \lambda_3 u_{xxx} + \lambda_4 (u u_x)_x + \lambda_5 u_{xx} = 0 \quad (4.17)$$

if we look for solution of (4.17) in the form of travelling wave (1.2).

Equation (4.17) was used in of works [48, 49, 50] for description of nonlinear waves. Some exact solutions were obtained in papers [28, 30].

4.2. Fourth order ODEs. Let us write the general form of the nonlinear four order ODEs with exact solution (4.1). It takes the form

$$a_0 y_{zzzz} + a_1 y y_{zz} + a_2 y_z^2 + a_3 y^3 + b_0 y_{zzz} + b_1 y y_z +$$

$$+ d_0 y_{zz} + d_1 y^2 + h_0 y_z - \alpha C_0 + C_1 = 0 \quad (4.18)$$

Substitution (4.1) into equation (4.12) at $A_2 = 1$, $A_1 = 0$, $A_0 = 0$ gives relations for parameters

$$a_3 = -120a_0 - 4a_2 - 6a_1 \quad (4.19)$$

$$b_1 = -12b_0 \quad (4.20)$$

$$d_1 = 240\alpha a_0 + 8\alpha a_2 - 6d_0 + 8\alpha a_1 \quad (4.21)$$

$$h_0 = 8\alpha b_0 \quad (4.22)$$

$$d_0 = 17\alpha a_0 + \frac{1}{2}\alpha a_2 - \frac{1}{8}C_0 + \frac{1}{4}\alpha a_1 \quad (4.23)$$

$$C_1 = -18\alpha^3 a_0 - \alpha^3 a_2 + \frac{1}{4}\alpha^2 C_0 - \frac{1}{2}\alpha^3 a_1 \quad (4.24)$$

We have nonlinear fourth order ODE with exact solution (4.1) at $A_0 = 0$, $A_1 = 0$ and $A_2 = 1$ in the form

$$\begin{aligned} a_0 y_{zzzz} + b_0 y_{zzz} + \left(a_1 y + 17 a_0 \alpha + \frac{1}{2} a_2 \alpha - \frac{1}{8} C_0 + 1/4 a_1 \alpha \right) y_{zz} + \\ + a_2 y_z^2 + (b_1 y + 8 b_0 \alpha) y_z + (-120 a_0 - 4 a_2 - 6 a_1) y^3 + \\ + \left(\frac{3}{4} C_0 + \frac{13}{2} a_1 \alpha + 5 a_2 \alpha + 138 a_0 \alpha \right) y^2 - C_0 \alpha y + C_1 = 0 \end{aligned} \quad (4.25)$$

Values α in equation (2.8) is found from equation (4.24) at given parameters C_1 , a_0 , a_2 and C_0 .

Assuming $a_0 = 1$, $a_1 = -20$, $a_2 = -10$, $a_3 = 40$, $b_0 = b_1 = d_0 = d_1 = h_0 = 0$ in equation (4.25) we have

$$y_{zzzz} - 20 y y_{zz} - 10 y_z^2 + 40 y^3 - \alpha C_0 y + C_1 = 0 \quad (4.26)$$

This is integrable equation. This one is found from the Korteweg – de Vries equation of the fifth order [41].

$$u_t + \frac{\partial}{\partial x} (u_{xxxx} - 20 u u_{xx} - 10 u_x^2 + 40 u^3) = 0 \quad (4.27)$$

Equation (4.26) has special solution (4.1) at $A_0 = A_1 = 0$, $A_2 = 1$ if we take $C_0 = 56\alpha$ and $C_1 = 16\alpha^3$.

Assuming $a_0 = 1$, $a_1 = -18$, $a_2 = -9$, $a_3 = 24$, $b_0 = b_1 = d_0 = d_1 = h_0 = 0$ in equation (4.25) we get

$$y_{zzzz} - 18 y y_{zz} + 24 y^3 - 9 y_z^2 - \alpha C_0 y + C_1 = 0 \quad (4.28)$$

This is integrable equation too. This one is found from the Schwarz – Kaup – Kupershmidt equation [51, 52]

$$u_t + \frac{\partial}{\partial x} (u_{xxxx} - 18 u u_{xx} - 9 u_x^2 + 24 u^3) = 0 \quad (4.29)$$

Equation (4.28) has special solution (4.1) at $A_0 = -2\alpha/3$, $A_1 = 0$ and $A_2 = 1$ if we take $C_0 = 0$ and $C_1 = -8\alpha^3/9$.

We can see again that our class of nonlinear ODEs contains a number of integrable equations.

Assuming $a_1 = 0$, $a_2 = 0$ and $b_1 = 0$ we have from equation (4.25)

$$\begin{aligned} a_0 y_{zzzz} + b_0 y_{zzz} + \left(-\frac{1}{8}C_0 + 17a_0\alpha\right) y_{zz} + 8b_0\alpha y_z - 120y^3 a_0 + \\ + \left(\frac{3}{4}C_0 + 138a_0\alpha\right) y^2 - C_0\alpha y + C_1 = 0 \end{aligned} \quad (4.30)$$

This equation can be obtained from nonlinear evolution equation in the form

$$\begin{aligned} u_t - 360a_0u^2u_x + \left(\frac{3}{2}C_0 + 276\alpha a_0\right) uu_x + \\ + 8\alpha a_0u_{xx} + \left(17\alpha a_0 - \frac{1}{8}C_0\right) u_{xxx} + b_0u_{xxxx} + a_0u_{xxxxx} = 0 \end{aligned} \quad (4.31)$$

if we look for solution using the travelling wave (1.2). Equation (4.31) were considered before. Exact solutions of this equation were found in works [11, 14, 53, 54].

5 Nonlinear ODEs with solution of the third degree singularity

Let us find nonlinear ODEs which have exact solution in the form

$$y(z) = A_0 + A_1Y + A_2Y^2 + A_3Y^3 \quad (5.1)$$

where $Y(z)$ is a solution of the Riccati equation too. One can see that $y(z)$ has the third degree singularity. We have not got any nonlinear second order ODE in the polynomial form with exact solutions (5.1). However we can find the nonlinear third order.

5.1. Third order ODEs. General form of the nonlinear differential equation with exact solution (5.1) is enough simple. It takes the form

$$a_0 y_{zzz} + a_1 y^2 + b_0 y_{zz} + d_0 y_z - 2C_0 y + C_1 = 0 \quad (5.2)$$

Using new variables [20] one can write equation (5.2) in the form

$$y_{zzz} + a_1 y^2 + \sigma y_{zz} + y_z - 2C_0 y + C_1 = 0 \quad (5.3)$$

Substituting solution (5.1) into equation (5.3) and taking into account equations (2.8), (2.5) and (2.6) we have

$$A_3 = 60, \quad a_1 = 1, \quad (5.4)$$

$$A_2 = -\frac{15}{2}\sigma, \quad (5.5)$$

$$A_1 = -60\alpha + \frac{30}{19} - \frac{15\sigma^2}{152} \quad (5.6)$$

$$A_0 = \frac{1}{2}C_0 - 5\sigma\alpha + \frac{7}{152}\sigma - \frac{13}{1216}\sigma^3 \quad (5.7)$$

We also have two additional equations

$$-\frac{75}{19}\alpha\sigma^2 + \frac{1200}{19}\alpha + \frac{1965}{11552}\sigma^4 + \frac{330}{361} - \frac{1305}{1444}\sigma^2 - 480\alpha^2 = 0 \quad (5.8)$$

and

$$\sigma(\sigma - 4)(\sigma + 4)(13\sigma^2 - 56 + 3040\alpha) = 0 \quad (5.9)$$

From equation (5.9) we get

$$\sigma_1 = 0, \quad \sigma_{2,3} = \pm 4, \quad (5.10)$$

$$\alpha = \frac{7}{380} - \frac{13}{3040}\sigma^2 \quad (5.11)$$

Substituting (5.10) and (5.11) into equation (5.8) we obtain

$$\alpha_1 = -\frac{1}{76}, \quad \alpha_2 = \frac{11}{76}, \quad \alpha_{3,4} = \pm \frac{1}{4} \quad (5.12)$$

$$\sigma_{4,5} = \pm \frac{16}{\sqrt{73}}, \quad \sigma_{6,7} = \pm \frac{12}{\sqrt{47}} \quad (5.13)$$

Using $\sigma_{4,5}$ and $\sigma_{6,7}$ we have from equation (5.11)

$$\alpha_5 = \frac{1}{292}, \quad \alpha_6 = \frac{1}{188} \quad (5.14)$$

As this take place constant C_1 takes

$$C_1^{(1)} = C_0^2 + \frac{225}{6859}, \quad C_1^{(2)} = C_0^2 - \frac{2475}{6859}, \quad C_1^{(3)} = C_0^2 - 9, \quad (5.15)$$

$$C_1^{(4)} = C_0^2 - 4, \quad C_1^{(5)} = C_0^2 - \frac{2025}{389017}, \quad C_1^{(6)} = C_0^2 - \frac{900}{103823}$$

As result of calculations we find six equations of equation (5.2) with exact solutions.

Assuming $\sigma = 0$ and $\alpha = -1/76$ we have equation

$$y_{zzz} + y^2 + y_z - 2yC_0 + \frac{225}{6859} + C_0^2 = 0 \quad (5.16)$$

with exact solution in the form

$$y(z) = C_0 + \frac{45}{19}Y + 60Y^3, \quad Y(z) = \pm \frac{1}{2\sqrt{19}} \operatorname{tg} \left(\pm \frac{z}{2\sqrt{19}} + \varphi_0 \right) \quad (5.17)$$

At $\sigma = 0$ and $\alpha = \alpha_2 = 11/76$ we obtain

$$y_{zzz} + y^2 + y_z - 2yC_0 - \frac{2475}{6859} + C_0^2 = 0 \quad (5.18)$$

with exact solution

$$y(z) = C_0 - \frac{135}{19}Y + 60Y^3, \quad Y(z) = \pm \frac{\sqrt{11}}{2\sqrt{19}} \tanh \left(\pm \frac{z\sqrt{11}}{2\sqrt{19}} + \varphi_0 \right) \quad (5.19)$$

Assuming $\sigma = 4$ and $\alpha = \alpha_3 = 1/4$ we get

$$y_{zzz} + y^2 + 4y_{zz} + y_z - 2yC_0 - 9 + C_0^2 = 0 \quad (5.20)$$

with exact solution

$$y(z) = \frac{9}{2} + C_0 - 15Y - 30Y^2 + 60Y^3, \quad Y(z) = \pm \frac{1}{2} \tanh \left(\pm \frac{z}{2} + \varphi_0 \right) \quad (5.21)$$

At $\sigma = 4$ and $\alpha = \alpha_4 = -1/4$ we obtain

$$y_{zzzz} + 4y_{zzz} + y_z + y^2 - 2C_0y + C_0^2 - 4 = 0 \quad (5.22)$$

with exact solution

$$y(z) = C_0 - \frac{11}{2} + 15Y - 30Y^2 + 60Y^3, \quad Y(z) = \pm \frac{1}{2} \operatorname{tg} \left(\pm \frac{z}{2} + \varphi_0 \right) \quad (5.23)$$

Assuming $\sigma = 16/\sqrt{73}$ and $\alpha = \alpha_5 = 16/\sqrt{73}$ we have

$$y_{zzz} + \frac{16}{\sqrt{73}}y_{zz} + y_z + y^2 - 2C_0y + C_0^2 = 0 - \frac{2025}{389017} \quad (5.24)$$

with exact solution

$$y(z) = C_0 + \frac{485\sqrt{73}}{63948} + \frac{1075}{1168}Y - \frac{120}{\sqrt{73}}Y^2 + 60Y^3 \quad (5.25)$$

$$Y(z) = \pm \frac{1}{2\sqrt{73}} \tanh\left(\pm \frac{z}{2\sqrt{73}} + \varphi_0\right)$$

At $\sigma = 12/\sqrt{47}$ and $\alpha = \alpha_6 = 12/\sqrt{47}$ we get

$$y_{zzz} + \frac{12}{\sqrt{47}}y_{zz} + y_z + y^2 - 2C_0y + C_0^2 - \frac{900}{103823} = 0 \quad (5.26)$$

with exact solution

$$y(z) = C_0 + \frac{45\sqrt{47}}{4418} + \frac{45}{47}Y - \frac{90\sqrt{47}}{47}Y^2 + 60Y^3, \quad (5.27)$$

$$Y(z) = \pm \frac{1}{2\sqrt{47}} \tanh\left(\pm \frac{z}{2\sqrt{47}} + \varphi_0\right)$$

All these equations can be found from the Kuramoto – Sivashinsky equation using the travelling wave (1.2). This equation takes the form [55, 56, 57, 58]

$$u_t + 2a_1uu_x + b_0u_{xxx} + d_0u_{xx} + a_0u_{xxxx} = 0 \quad (5.28)$$

Solutions of this equation were first found in work [11] but they were re-discovered a few times later. At $\sigma = 4$ there is periodic solution of equation (5.3) [13].

We found that there is only one form of the nonlinear third order differential equation with exact solution of the third degree singularity.

5.2. Fourth order ODEs. General form of nonlinear fourth order ODEs with solutions (5.1) can be presented as the following

$$a_0y_{zzzz} + a_1yy_z + b_0y_{zzz} + b_1y^2 + d_0y_{zz} + h_0y_z - C_0\alpha y + C_1 = 0 \quad (5.29)$$

For more simple calculations let us first take $A_2 = 0$ and $A_3 = 120$. We have

$$a_1 = a_0 \quad (5.30)$$

$$b_1 = \frac{1}{2}b_0 \quad (5.31)$$

$$d_0 = \frac{19}{60}a_0A_1 + 38\alpha a_0 \quad (5.32)$$

$$h_0 = \frac{19}{60}b_0A_1 + 38\alpha b_0 - a_0A_0 \quad (5.33)$$

$$A_0 = \frac{7}{5} \frac{a_0A_1\alpha}{b_0} + 108 \frac{a_0\alpha^2}{b_0} + \frac{C_0\alpha}{b_0} + \frac{11a_0A_1^2}{3600b_0}, \quad b_0 \neq 0 \quad (5.34)$$

We also have two values for α

$$\alpha_1^{(1)} = -\frac{A_1}{360}, \quad \alpha_2^{(1)} = -\frac{11A_1}{1080} \quad (5.35)$$

As this takes place we obtain two values of the constant C_1

$$C_1^{(1)} = \frac{121}{2332800} \frac{A_1^2 C_0^2}{b_0} + \frac{11}{43740} A_1^3 b_0 \quad (5.36)$$

$$C_1^{(2)} = \frac{1}{259200} \frac{A_1^2 C_0^2}{b_0} + \frac{1}{1620} A_1^3 b_0 \quad (5.37)$$

Equation with exact solution

$$y(z) = -\frac{1}{360} \frac{A_1 C_0}{b_0} + A_1 Y + 120 Y^3 \quad (5.38)$$

takes the form

$$\begin{aligned} a_0 y_{zzzz} + b_0 y_{zzz} + d_0 y_{zz} - \frac{(360 a_0 y b_0 + 76 A_1 b_0^2 + A_1 a_0 C_0) y_z}{360 b_0} + \\ + \frac{1}{2} b_0 y^2 + \frac{1}{360} A_1 C_0 y + C_1 = 0 \end{aligned} \quad (5.39)$$

Where C_1 is determined by formulas (5.36) and (5.37).

Assuming $b_0 = 0$ we have $A_0 \neq 0$ but we find C_0 in the form

$$C_0 = -\frac{a_0 (5040 A_1 \alpha + 388800 \alpha^2 + 11 A_1^2)}{3600 \alpha} \quad (5.40)$$

In this case we have

$$\alpha_1^{(2)} = -\frac{11A}{1080}, \quad \alpha_2^{(2)} = -\frac{A_1}{360}, \quad \alpha_3^{(2)} = -\frac{A_1}{120} \quad (5.41)$$

and three values of constant C_1 in the form

$$C_1^{(1)} = 0, \quad C_1^{(2)} = 0, \quad C_1^{(3)} = \frac{a_0 A_0 A_1^2}{900} \quad (5.42)$$

Equations with exact solution

$$y(z) = A_0 + A_1 Y + 120 Y^3 \quad (5.43)$$

can be presented in the form

$$a_0 y_{zzzz} + a_0 y y_z - \frac{19a_0 A_1}{270} y_{zz} - a_0 A_0 y_z = 0 \quad (5.44)$$

$$a_0 y_{zzzz} + a_0 y y_z + \frac{19a_0 A_1}{90} y_{zz} - a_0 A_0 y_z = 0 \quad (5.45)$$

$$a_0 y_{zzzz} + a_0 y y_z - a_0 A_0 y_z - \frac{a_0 A_1^2}{900} y + C_1 = 0 \quad (5.46)$$

We have not got any information about physical application of equations (5.44), (5.45) and (5.46).

6 Nonlinear ODEs with exact solutions of the fourth order singularity

Let us find nonlinear ordinary differential equations which have exact solutions of the fourth order singularity. These solutions can be presented by the formula

$$y(z) = A_0 + A_1 Y + A_2 Y^2 + A_3 Y^3 + A_4 Y^4 \quad (6.1)$$

where $Y(z)$ satisfies the Riccati equation again.

We can not suggest nonlinear ODEs of the second and third order of the polynomial form with solution (6.1). In this case we can take the fourth order ODE in the form

$$y_{zzzz} + a_1 y^2 + b_0 y_{zzz} + d_0 y_{zz} + e_0 y_z - C_0 y + C_1 = 0 \quad (6.2)$$

For calculations it is convenient to use first $A_2 = 0$, $A_3 = 840\beta/11$ (β is new parameter), $A_4 = -840n$. Substituting (6.1) into equation (6.2) we have

$$a_1 = 1 \quad b_0 = \beta, \quad d_0 = \frac{9}{121} \beta^2 + 104 \alpha \quad (6.3)$$

$$e_0 = -\frac{27}{1331}\beta^3 + \frac{574}{11}\beta\alpha + \frac{69}{140}A_1 \quad (6.4)$$

$$A_0 = \frac{C_0}{2} - \frac{90}{121}\beta^2\alpha + \frac{81}{29282}\beta^4 + \frac{31}{1540}A_1\beta + 816\alpha^2 \quad (6.5)$$

$$A_1 = 443520 \frac{\beta\alpha^2}{-30008\alpha + 65\beta^2} + \frac{32760}{11} \frac{\beta^3\alpha}{-30008\alpha + 65\beta^2} - \frac{11340}{1331} \frac{\beta^5}{-30008\alpha + 65\beta^2} \quad (6.6)$$

We also have equation for α in the form

$$\begin{aligned} & (-12100\alpha + 27\beta^2)(-484\alpha + \beta^2) \times \\ & \times (81\beta^6 - 696960\beta^4\alpha + 289774672\beta^2\alpha^2 + 7029554048\alpha^3) = 0 \end{aligned} \quad (6.7)$$

Assuming

$$\alpha = \frac{1}{484}\beta^2 \quad (6.8)$$

we also have the constant C_1 in the form

$$C_1 = -\frac{900}{214358881}\beta^8 + \frac{C_0^2}{4} \quad (6.9)$$

In this case we get equation

$$\begin{aligned} & y_{zzzz} + \beta y_{zzz} + \frac{35}{121}\beta^2 y_{zz} + \frac{13}{1331}\beta^3 y_z + y^2 - \\ & - C_0 y - \frac{900}{214358881}\beta^8 + \frac{C_0^2}{4} = 0 \end{aligned} \quad (6.10)$$

with exact solution

$$y(z) = C_0 + \frac{45}{29282}\beta^4 - \frac{210}{1331}\beta^3 Y + \frac{840}{11}\beta Y^3 - 840 Y^4 \quad (6.11)$$

where

$$Y(z) = \pm \frac{\beta}{22} \tanh\left(\pm \frac{\beta z}{22} + \varphi_0\right) \quad (6.12)$$

Assuming $A_1 = 0$, $A_2 = 140n$, $A_3 = 0$ and $A_4 = -840$ we have after calculations

$$a_1 = 1, \quad b_0 = 0, \quad d_0 = -39n + 104\alpha, \quad e_0 = 0, \quad (6.13)$$

$$A_0 = \frac{C_0}{2} + 816\alpha^2 + \frac{93}{2}n^2 - 528n\alpha$$

We get three values for α

$$\alpha_1 = \frac{n}{4}, \quad \alpha_{2,3} = \frac{1}{4} \left(\frac{31}{16} \pm \frac{i\sqrt{31}}{16} \right) \quad (6.14)$$

and the constant C_1 in the form

$$C_1 = -324n^4 + \frac{C_0^2}{4} \quad (6.15)$$

In this case we obtain equation in the form

$$y_{zzzz} + y^2 - 13ny_{zz} - C_0y - 324n^4 + \frac{C_0^2}{4} = 0 \quad (6.16)$$

with exact solution

$$y(z) = \frac{C_0}{2} - \frac{69}{2}n^2 + 420nY^2 - 840Y^4 \quad (6.17)$$

where

$$Y(z) = \pm \frac{\sqrt{n}}{2} \tanh \left(\pm \frac{\sqrt{n}z}{2} + \varphi_0 \right) \quad (6.18)$$

Exact solutions of equation (6.16) were found in works [13, 14] and rediscovered in a number of papers later.

7 Nonlinear ODEs with exact solution of the fifth degree singularity

Assume that nonlinear ODEs have exact solution of the fifth degree singularity. Simplest case of this solution takes the form

$$y(z) = A_0 + A_1Y + A_2Y^2 + A_3Y^3 + A_4Y^4 + A_5Y^5 \quad (7.1)$$

One can see that nonlinearity for equation with solution (7.1) is y^2 . We have term of the tenth degree singularity of this equation. If we want to have the polynomial form of this equation we have to take

$$y_{zzzzz} + a_1 y^2 = 0 \quad (7.2)$$

However we can add to equation (7.2) other terms with lesser singularity. As a result we have nonlinear ODE in the form

$$y_{zzzzz} + b_0 y_{zzzz} - d_0 y_{zzz} + e_0 y_{zz} + h_0 y_z + a_1 y^2 - C_0 y + C_1 = 0 \quad (7.3)$$

This discussion were given by N.A. Kudryashov in the work [26] to look for the truncated expansions for nonlinear ODE with the simplest nonlinearity.

Equation (7.3) is found from nonlinear evolution equation which takes the form

$$u_t + 2a_1 u u_x + h_0 u_{xx} + e_0 u_{xxx} - d_0 u_{xxxx} + b_0 u_{xxxxx} + u_{xxxxxx} = 0 \quad (7.4)$$

Last years equation (7.4) was used for the description of the chaos model [59, 60, 61, 62] and it is important to have exact solutions of this equation. Let us find exact solutions of equation (7.3) at $e_0 = 0$ and $b_0 = 0$.

Using new variables

$$z' = zL^{-1}, \quad y' = yB^{-1}, \quad L = d_0^{-1/2}, \quad B = d_0^{5/2} a_1^{-1}, \quad (7.5)$$

$$C'_0 = C_0 d_0^{-5/2}, \quad C'_1 = C_1 d_0^{-5/2} B^{-1}$$

We have equation from (7.3) in the form

$$y_{zzzzz} - y_{zzz} + \sigma y_z + \frac{1}{2} y^2 - C_0 y + C_1 = 0 \quad (7.6)$$

(primes are omitted in equation (7.6)).

Substituting (7.1) into equation (7.6) we obtain

$$A_5 = 30240, \quad A_4 = 0, \quad A_3 = -\frac{2520}{11} - 50400\alpha, \quad A_2 = 0, \quad (7.7)$$

$$A_1 = \frac{1260}{251}\sigma + 20160\alpha^2 - \frac{12600}{30371} + \frac{2520}{11}\alpha, \quad A_0 = C_0$$

We also have

$$\sigma = \frac{-92400\alpha + 10204656\alpha^2 + 213811840\alpha^3 + 2045}{121(9240\alpha - 79)} \quad (7.8)$$

Taking into account (7.8) we get six values for α

$$\alpha_1 = \frac{1}{440}, \quad \alpha_2 = \frac{5}{176}, \quad \alpha_3 = \frac{1}{220},$$

$$\alpha_4 = \frac{46031}{52800m} - \frac{557}{52800} - \frac{m}{52800}, \quad (7.9)$$

$$\alpha_{5,6} = \frac{m}{105600} - \frac{46031}{105600m} - \frac{557}{52800} \pm \frac{i\sqrt{3}}{88} \left(-\frac{m}{1200} - \frac{46031}{1200m} \right)$$

Where

$$m = \left(113816753 + 1260\sqrt{8221079733} \right)^{\frac{1}{3}} \quad (7.10)$$

Assuming $\alpha = \alpha_1$ we have equation

$$y_{zzzzz} - y_{zzz} + \frac{3259}{12100} y_z + \frac{1}{2} y^2 - C_0 y - \frac{321489}{322102000} + \frac{1}{2} C_0^2 = 0 \quad (7.11)$$

with exact solution

$$y(z) = C_0 + \frac{189}{121} Y - \frac{3780}{11} Y^3 + 30240 Y^5 \quad (7.12)$$

where

$$Y(z) = \pm \frac{1}{2\sqrt{110}} \tanh \left(\pm \frac{z}{2\sqrt{110}} + \varphi_0 \right)$$

At $\alpha = \alpha_2$ we obtain

$$y_{zzzzz} - y_{zzz} - \frac{1095}{1936} y_z + \frac{1}{2} y^2 - C_0 y - \frac{12403125}{82458112} + \frac{1}{2} C_0^2 = 0 \quad (7.13)$$

with exact solution

$$y(z) = C_0 + \frac{4725}{242} Y - \frac{18270}{11} Y^3 + 30240 Y^5 \quad (7.14)$$

where

$$Y(z) = \pm \frac{\sqrt{15}}{16\sqrt{11}} \tanh \left(\pm \frac{z\sqrt{5}}{16\sqrt{11}} + \varphi_0 \right) \quad (7.15)$$

Assuming $\alpha = \alpha_3$ we have equation

$$y_{zzzzz} - y_{zzz} + \frac{114}{275} y_z + \frac{1}{2} y^2 - 2 C_0 y - \frac{127008}{20131375} + 2 C_0^2 = 0 \quad (7.16)$$

with exact solution in the form

$$y(z) = C_0 + \frac{378}{121} Y - \frac{5040}{11} Y^3 + 30240 Y^5, \quad (7.17)$$

$$Y(z) = \pm \frac{1}{2\sqrt{55}} \tanh \left(\pm \frac{z}{2\sqrt{55}} + \varphi_0 \right)$$

We hope these exact solutions will be useful at the study of the turbulence processes where equation (7.4) is used.

8 Conclusion

Let us emphasize in brief the results of this work. We noted that a lot of nonlinear differential equations have exact solutions expressed via hyperbolic functions. It is well known that the base of the hyperbolic functions is the general solution of the Riccati equation. The widely used tanh method applied for finding exact solutions of many nonlinear differential equations confirms this idea.

This observation suggested that a lot of special solutions of nonlinear differential equations can be presented in the form of the general solution of the Riccati equation. Using this observation we formulated new problem that is to find nonlinear ODEs with exact solutions. We have found a number of nonlinear differential equations of the second, third and fourth order which have exact solutions. Exact solutions of these equations have different singularities and are expressed via general solutions of the Riccati equation. We also list a number of nonlinear ODEs with exact solutions that are found from the widely used nonlinear evolution equations.

9 Acknowledgments

This work was supported by the International Science and Technology Center under Project No 1379-2. This material is partially based upon work supported by the Russian Foundation for Basic Research under Grant No 01-01-00693.

References

- [1] Musette M. and Conte R., 2003, *Physica D*, **181**, 70-79
- [2] Liu S.K., Fu Z.T., Liu S.D. et al., 2003, *Phys. Lett. A.*, **309**, 234
- [3] Yan Z.Y., 2003, *Chaos Solitons Fractals*, **15**, 575
- [4] Yan Z.Y., 2003, *Chaos Solitons Fractals*, **15**, 891
- [5] Elwakil S.A., Ellabany S.K., Zahran M.A. et al., 2002, *Phys. Lett. A.*, **299**, 179
- [6] Fan E.G., 2001, *Nuovo Cimento B*, **116**, 1385
- [7] Fu Z.T., Liu S.K., Liu S.D. et al., 2001 *Phys. Lett. A.*, **290**, 72
- [8] Fan E.G., Chao L., 2001, *Phys. Lett. A.*, **285**, 373
- [9] Kudryashov N.A., Soukharev M.B., 2001, *J. Appl. Math. Mech*, **65**, 855
- [10] Conte R. (ed) 1999, The Painleve approach to nonlinear ordinary differential equations *The Painleve Property. One Century Later (CRM Series in Mathematics Physics)* (Berlin: Springer)
- [11] Kudryashov N.A., 1988, *Journal of Applied Mathematics and Mechanics*, **52**, 361-365
- [12] Conte R. and Musette M., *J. Phys. A.*, 1989, **22**, 169-177
- [13] Kudryashov N.A., 1990, *Phys Lett. A.*, **147**, 287-291
- [14] Kudryashov N.A., 1991, *Phys Lett. A.*, **155**, 269-275
- [15] Berloff N.G., Howard L.N., 1997, *Stud. Appl. Math.*, **99**, 1-24
- [16] Berloff N.G., Howard L.N., 1998, *Stud. Appl. Math.*, **100**, 195-213
- [17] Kudryashov N.A., *Analytical theory of nonlinear differential equations*, Moscow-Igevsik, IKI (2003) 360 p
- [18] Choudhury S.R., 1997, *Phys. Lett. A.*, **159**, 311-317
- [19] Conte R., Musette M., 1992, *J. Phys. A: Math. Gen.* **25**, 5609-5623
- [20] Kudryashov N.A., and Zargaryan E.D. 1996, *J. Phys. A. Math. and Gen* **29**, 8067-8077
- [21] Ince F.L., 1956, *Ordinary Differential Equations* (New York, Dover)
- [22] Weiss J., Tabor M. and Carnevale G., 1983, *J. Math. Phys.* **24**, 522-526
- [23] Pickering A., 1993, *J. Phys. A: Math. Gen.*, **26**, 288-300
- [24] Fan E. G., 2000, *Phys Lett. A.*, **227**, 212-218
- [25] Elwakil S.A., Ellabany S.K., Zahran M.A. et al., 2003, *Z. Naturforsch*, **58**, 39-44

- [26] Kudryashov N.A., 1989, Mathematical simulation, **1**, No 9, 151-158 (in Russian)
- [27] Kudryashov N.A., 1989, Mathematical simulation, **1**, No 6, 55 (in Russian)
- [28] Lou S.Y., Huang G.X., Ruan. H.Y., 1991, J. phys. A.: Math. Gen., **24**, L587-L590
- [29] Parkes E.J., Duffy B.R., 1996, Comput. Phys. Commun., **3** 288-300
- [30] Porubov A., 1993, J. Phys. A.: Math. Gen., **26**, L797-L800
- [31] Malfliet W., Hereman W., 1996, Phys. scripta, **54**, 563-568
- [32] Malfliet W., 1993, J. Phys. A.: Math. Gen., **26** L723-L728
- [33] Conzalez A. and Castellanos A., 1994, Phys. Rev. E., **49**, 2935-2940
- [34] Jeffrey A. and Xu S., 1989, Wave motion, **11**, 559
- [35] Berkovich L.M. Factorization and transformation of differential equations, 2002, RHD, Moscow, 464p. (in Russian)
- [36] Gudkov V.V. 1995, Journal of numerical mathematics and mathematical physics, **35**, 615-623 (in Russian)
- [37] Fisher R.A., 1937, Annals of Eugenics, **7**, 355-369
- [38] Kolmogorov A.N., Petrovskii I.G., Piskunov N.S., 1937, Bulletin MSU, **1**, 1-26
- [39] Ablowitz M.J. and Zeppetella A., 1979, Bull. Math. Biol., **41**, 835-840
- [40] Kudryashov N.A., 1994, J. Phys. A.: Math. Gen., **27**, 2457-2470
- [41] Ablowitz M.J. and Clarkson P.A., 1991, Solitons, Nonlinear Evolution Equations and inverse Scattering, Cambridge university press, 516 p.
- [42] Fordy A.P. and Gibbons J., 1980, Phys. Lett. A., **160**, 347
- [43] Hone A.W., 1980, Physica D., **118**, 1-16
- [44] Bar D.E., Nepomnyaschy A.A., 1995, Physica D., **86**, 586-602
- [45] Yamamoto Y., Takizawa E.I., 1981, J. Phys. Soc. Jpn., **50**, 1055-602
- [46] Kano K., Nakayama T., 1981, J. Phys. Soc. Jpn., **50**, 361
- [47] Olver V.I., 1984, Hamiltonian and non-Hamiltonian models for water waves, in Lecture notes in Physics, No. (Springer - Verlag, New York), 273-290
- [48] Aspe H. and Depassier M.C., 1990, Phys. Rev. A., **41**, 3125
- [49] Kawahara T and Toh S., 1988, Phys. Fluids, **31**, 2103
- [50] Garazo A. and Velarde M.G., 1991, Phys. Fluids A, **3**, 2295
- [51] Weiss J., 1984, J. Math. Phys. **25**, 13-24

- [52] Kudryashov N.A., 1999, J. Phys. A.: Math. Gen., **32**, 999-1013
- [53] Feng B.F., Kawahara T., 2002, J. bifurcat. Chaos, **12**, 2393-2407
- [54] Zemlyanukhin A.I., Mogilevich L.I., 2001 Acoust. Phys., **47**, 303-307
- [55] Kuramoto Y. and Tsuzuki T., 1976 Prog. Theor. Phys, **55**, 356
- [56] Sivashinsky G.I., 1982, **4**, 227-235
- [57] Kawahara T., 1983, Phys. Rev. Lett, **51**, 381
- [58] Alfaro C.M., Benguria R.D. and Depassier M.C., 1992, Physica D, **61**, 1-5
- [59] Beresnev L.A. and Nikolaevskiy V.N., 1993, Physica D, **66**, 1-6
- [60] Tribelsky M.I. and Tsuboi K., 1996, Phys. Rev. Lett., **76**, 1631
- [61] Xi H.-W., Toral R., Gunton J.D. and Tribelsky M.I., 2000, Phys. Rev E., **62**, 17-20
- [62] Toral R., Xiong G., Gunton J.D. and Xi H.-W., 2003, J.Phys.A. Math. Gen., **36**, 1323-1335